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AN ALGORITHM FOR THE
MIXED INTEGER PROBLEM

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AN ALGORITHM FOR THE MIXED INTEGER PROBLEM

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SUMMARY

An algorithm is given for the numerical solution of the "mixed integer" linear programming problem, the problem of maximizing a linear form in finitely many variables constrained both by linear inequalities and the requirement that a proper subset of the variables assume only integral values. The algorithm is an extension of the cutting plane technique for the solution of the "pure integer" problem.

AN ALGORITHM FOR THE MIXED INTEGER PROBLEM

Ralph Gomory

The problem discussed here is an integer programming problem, i.e., the problem of maximizing

$$z = a_{o,o} + \sum_{j=1}^{j=n} a_{o,j}(-t_j).$$

subject to the inequalities

$$(1) \qquad \sum_{j=1}^{j=n} a_{i,j}t_j \leq a_{i,o}, \qquad i = 1, \ldots, m$$

and subject to the additional condition that some specified subcollection of the variables appearing above should be integers.

If the inequalities above are changed into equations in nonnegative variables by the addition of m "slack" variables, and the whole set is enlarged to form a set in which <u>all</u> the variables are expressed in terms of the independent or "nonbasic" ones, we have

$$z = a_{o,o} + \sum_{j=1}^{j=n} a_{o,j}(-t_j)$$

$$s_i = a_{i,o} + \sum_{j=1}^{j=n} a_{i,j}(-t_j) \qquad i = 1, \ldots, m$$

$$t_j = -1(-t_j) \qquad j = 1, \ldots, n.$$

For the sake of a more uniform notation we will rewrite this as

$$(2) \qquad x_i = a_{i,o} + \sum_{j=1}^{j=n} a_{i,j}(-t_j) \qquad i = 0, \ldots, m+n,$$

where the x_i now are <u>all</u> the variables and the $a_{i,j}$ are <u>all</u> the coefficients.

The usual[1] linear programming problem is solved by applying G. B. Dantzig's simplex method. In this method a series of "pivot steps," "Gaussian eliminations," "changes of basis," or "changes to different sets of nonbasic variables" bring the equations (2) into a form in which, denoting the new coefficients in the equations by primes,

$$(i) \quad a'_{i,o} \geq 0 \qquad i = 1, \ldots, m+n$$

and

$$(ii) \quad a'_{o,j} \geq 0 \qquad j = 1, \ldots, n.$$

The first condition is the condition that in the "trial solution" obtained by putting all the nonbasic variables equal to zero, the values that result for all the variables are nonnegative. The second condition makes certain that the objective function is in fact maximal when the variables are given the values they attain in this trial solution. The solution obtained is of course

[1] The usual method terminates when conditions (ii) first hold. It is necessary here that the pivoting continue until all columns $j > 0$ become lexicographically positive. The procedure for doing this is described in [1].

$$x_i = a'_{i,o} \qquad i = 0, \ldots, m+n.$$

This solution may very well not satisfy the integer requirement, i.e., some x_i that is required to be an integer is assigned the noninteger value $a'_{i,o}$.

If this occurs we will be able to deduce a new inequality that will be satisfied by any integer solution, i.e., by any solution having integers where they are required, but will not be satisfied by the current trial solution.

Then, just as in [1] and [2], this new inequality will be added to the original set of inequalities, and the new set then remaximized by the simplex method. This remaximization is usually quite rapid as adding the new inequality maintains dual feasibility, and introduces just the one unsatisfied inequality.

If the new maximum solution still contains integer variables which are assigned noninteger values the process is repeated.

To deduce this new inequality we make use of the equation

$$(3) \qquad x_i = a'_{i,o} + \Sigma\, a'_{i,j}(-t_j)$$

where the x_i is an integer variable, $a'_{i,o}$ is noninteger, and the t_j are the current set of nonbasic variables. Since $a'_{i,o}$ is noninteger it can be written uniquely as the sum of an integer $n'_{i,o}$ and a fractional part $f'_{i,o}$, $0 < f'_{i,o} < 1$.

We now imagine that we have an integer solution to the problem and use x_i', t_j' to denote the values given to the variables in (3) by this solution. Hence

$$x_i' = a_{i,o}' + \Sigma\, a_{i,j}'(-t_j')$$

and using $a \equiv b$ to mean a and b differ by an integer (equivalence modulo 1), we have, since $x_i' \equiv 0$ and $a_{i,o}' \equiv f_{i,o}'$,

$$\Sigma\, a_{i,j}' t_j' \equiv f_{i,o}'. \tag{4}$$

We will group the constants on the left in (4) according to their sign. Let S^+ be the set of indices j for which $a_{i,j}' \geq 0$, and S^- the set for which $a_{i,j}' < 0$. Then

$$\sum_{j\in S^+} a_{i,j}' t_j' + \sum_{j\in S^-} a_{i,j}' t_j' \equiv f_{i,o}'. \tag{5}$$

There are now two possibilities to consider. Either the expression on the left is (i) nonnegative, or (ii) negative.

Case (i). Since the left side is nonnegative and equivalent to f_o, its value can only be f_o', or $1 + f_o'$, or $2 + f_o'$, etc. Hence

$$f_{i,o}' \leq \sum_{j\in S^+} a_{i,j}' t_j' + \sum_{j\in S^-} a_{i,j}' t_j' \leq \sum_{j\in S^+} a_{i,j}' t_j'.$$

Case (ii). If the right-hand side is negative and equivalent to $f'_{i,o}$ it can only be $f_o - 1$, $f_o - 2$, etc. So in every case

$$f'_{i,o} - 1 \geq \sum_{j \in S^+} a'_{i,j} t'_j + \sum_{j \in S^-} a'_{i,j} t'_j \geq \sum_{j \in S^-} a'_{i,j} t'_j,$$

or, multiplying by $-f'_{i,o}/1 - f'_{i,o}$,

$$f'_{i,o} \leq \sum_{j \in S^-} \frac{f'_{i,o}}{1 - f'_{i,o}} (-a'_{i,j}) t'_j .$$

Now either (i) holds or (ii) holds so always

$$(6) \qquad f'_{i,o} \leq \sum_{j \in S^+} a'_{i,j} t'_j + \sum_{j \in S^-} \frac{f'_{i,o}}{1 - f'_{i,o}} (-a'_{i,j}) t'_j,$$

since the right side is the sum of two nonnegative numbers, one of which is $\geq f'_{i,o}$.

This inequality then is satisfied by any integer solution but not by the present trial solution since substituting $t_j = 0$ for all j into (6), makes the right-hand side 0.

Of course the inequality (6) can be rewritten as an equation by introducing a nonnegative slack s. Then (6) becomes

$$s = -f'_{i,o} - \sum_{j \in S^+} a'_{i,j}(-t_j) - \sum_{j \in S^-} \frac{f'_{i,o}}{1 - f'_{i,o}} (-a'_{i,o})(-t_j)$$

In obtaining (6) we have used only the fact that x_i was required to be an integer. If some of the nonbasic variables t_j are also integer variables, the inequality (6) can be improved in a manner entirely analogous to the reduction that is always possible in the strictly integer problem. The improvement will take the form of a decrease in the coefficients on the right in the resulting inequality (6). It is clear that for fixed $f'_{i,o}$ the smaller these coefficients, the stronger the inequality.

Let us suppose then that some t_{j_o} is required to be integer and hence is assigned an integer value t'_{j_o} in (5). Changing a'_{i,j_o} by an integer amount then changes the left side of (5) by an integer, and hence preserves the equivalence. Thus we may replace a'_{i,j_o} by any new value $a^* \equiv a_{i,j_o}$ and proceed just as before to deduce an inequality like (6).

If $a^* \geq 0$, the coefficient of t_{j_o} in the resulting inequality is simply a^*. If a^* is < 0, it is $-f'_{i,o}/(1 - f'_{i,o})a^*$. Among $a^* \geq 0$, $a^* = f'_{i,j_o}$ the fractional part[1] of a'_{i,j_o} clearly gives the smallest coefficient to t_{j_o} in the resulting inequality. (This may even be 0.) Among $a^* < 0$, the smallest coefficient is obtained from $a^* = f'_{i,j_o} - 1$, and is

$$(7) \qquad \frac{f'_{i,o}}{1 - f_{i,o}} (1 - f'_{i,j_o}).$$

[1] By the fractional part of both positive and negative numbers $a_{i,j}$ we will mean the nonnegative fraction $f_{i,j} < 1$ such that $a_{i,j} = n_{i,j} + f_{i,j}$ with $n_{i,j}$ integer.

To obtain the smallest possible coefficient we choose the smaller of f'_{i,j_o} and (7) which, because an expression of the form $x/1 - x$ increases monotonically with x is seen to be

$$f'_{i,j_o} \text{ if } f'_{i,j_o} \leq f'_{i,o}$$

and

$$\frac{f'_{i,o}}{1 - f_{i,o}} \left(1 - f'_{i,\,j_o}\right) \text{ if } f'_{i,j_o} > f'_{o,o}.$$

It follows that the strongest inequality is obtained by a simple two-stage process. (i) First replace coefficients of integer variables by their fractional parts if these are less than $f'_{i,o}$, or by the fractional parts less 1 if they are greater than $f'_{i,o}$. (ii) Then deduce the inequality (6) as before. The final result obtained from the equation

$$x_i = a'_{i,o} + \Sigma\, a'_{i,j}(-t_j)$$

by this procedure is the inequality represented by the equation

$$s = -f'_{i,o} - \Sigma\, f^*_{i,j}(-t_j) \tag{8}$$

where the $f^*_{i,j}$, all nonnegative, are given by the following formulae:

$$f^*_{i,j} = \begin{cases} a'_{i,j} & \text{if } a'_{i,j} \geq 0 \text{ and } t_j \text{ noninteger variable} \\ \dfrac{f'_{i,o}}{1 - f'_{i,o}} (-a'_{i,j}) & \text{if } a'_{i,j} < 0 \text{ and } t_j \text{ noninteger variable} \\ f'_{i,j} & \text{if } f'_{i,j} \leq f'_{i,o} \text{ and } t_j \text{ integer variable} \\ \dfrac{f'_{i,o}}{1 - f'_{i,o}} (f'_{i,j} - 1) & \text{if } f'_{i,j} > f'_{i,o} \text{ and } t_j \text{ integer variable} \end{cases}$$

Equation (8) is then added and the problem is remaximized. It seems sensible to use the dual simplex method at this point as all the $a'_{o,j}$, $j \geq 1$, are nonnegative, and there is only one negative element, $-f'_{i,o}$, in the 0-column.

If the dual simplex method is applied, the 0-column is decreased lexicographically at the next step, and furthermore, denoting by double primes the coefficients after the next pivot step and by j_o the column in which the pivot step takes place, we have

$$(9) \qquad \begin{aligned} a''_{i,o} &\leq n'_{i,o} && \text{if } a'_{i,j_o} > 0 \\ a''_{i,o} &\geq n'_{i,o} + 1 && \text{if } a'_{i,j_o} < 0 \end{aligned}$$

where $n'_{i,o}$ is the integer part of $a'_{i,o}$, the index i in (9) is that of the row figuring in equations (3) through (8).

This means that after the next pivot step the value assigned to x_i by the new trial solution is either $\geq$ the next highest integer, or $\leq$ the next lowest integer.

To see this we consider the mechanism of the dual simplex method. The dual simplex method will pick a pivot in the new row represented by (8). If the pivot element is chosen in this row and in the j_o column then the formula for the $a''_{i,o}$ that results after a pivot step is

$$a''_{i,o} = a'_{i,o} - \frac{f'_{i,o}a_{i,j_o}}{f^*_{i,j_o}} .$$

Now the formulas for $f'_{i,j}$ show that if a_{i,j_o} is positive and t_{j_o} noninteger we have

$$a''_{i,o} = a'_{i,o} - \frac{f'_{i,o}a'_{i,j_o}}{a'_{i,j_o}} = n'_{i,o} . \tag{10}$$

If a'_{i,j_o} is negative and t_{j_o} noninteger we have

$$a''_{i,o} = a'_{i,o} - \frac{f'_{i,o}a'_{i,j_o}}{\left(\frac{f'_{i,o}}{1-f'_{i,o}}\right)\left(-a'_{i,j_o}\right)} \tag{11}$$

$$= a'_{i,o} - f'_{i,o} + 1 = n'_{i,o} + 1 .$$

To cover the cases when t_{j_o} is an integer variable we need only remember that in this case the f^*_{i,j_o} is deduced by a two-stage process, part (ii) of which is exactly the same as the process used to deduce the f^*_{i,j_o} when t_{j_o} is noninteger. Consequently if part (i) leaves $a'_{i,j}$ unchanged, either (10) or (11) holds just as above. Part (i) will have $a'_{i,j}$ unchanged only if either

$$a'_{i,j} = f'_{i,j}, \qquad \text{and } f'_{i,j} \leq f'_{i,o}$$

or

$$a'_{i,j} < 0,\ a'_{i,j} = f'_{i,j} - 1, \qquad \text{and } f'_{i,j} > f'_{i,o}$$

Otherwise part (i) makes a change which results in a strictly smaller final $f^*_{i,j}$. So in these cases we have the strict inequalities

$$a''_{i,o} < n'_{i,o} \qquad \text{if } a'_{i,j_o} > 0$$

$$a''_{i,o} > n'_{i,o} + 1 \qquad \text{if } a'_{i,j_o} < 0.$$

The remaining possibility, $a'_{i,j_o} = 0$, can not occur because $a'_{i,j_o} = 0$ implies $f^*_{i,j_o} = 0$ and so f^*_{i,j_o} can not be the pivot element. Thus (9) holds in all cases.

Now (9) is exactly the property required for a finiteness proof—i.e., a proof that the solution is attained in a finite number of steps—provided that the objective function z is one of the integer variables. To see this we arrange the original

equations so that the integer variables on the left in (2) are the first rows following the objective function z. (This means that they rank higher lexicographically in the dual simplex method.) Given property (9), the reasoning in the first finitness proof in [1] (pp. 33–35) now goes through unchanged. Of course one must stop now on attaining the required integer values in the 0–column, as an all–integer matrix is not generally obtained.

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